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Additional solutions of a weighted heat equation

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ABSTRACT

We obtain a simple algorithm for computing additional solutions of a weighted heat equation.

1. Introduction

We recall Legendre's differential equation [1]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}v(x)}{\mathrm{d}x}\right]+n(n+1)v(x)=0 \tag{1.1}$$

which has the Legendre polynomial given by Rodrigues' formula:

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n[(x^2-1)^n]}{\mathrm{d}x^n} \tag{1.2}$$

as one of its solutions.

The series representation is

$$P_n(x)=\sum_{k=0}^{[n/2]}\frac{(-1)^k(1/2)_{n-k}(2x)^{n-2k}}{k!(n-2k)!} \tag{1.3}$$

with the generating function

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n. \tag{1.4}$$

Expanding the Taylor series in Eq. (1.4) for the first two terms gives

$$P_0(x)=1,\qquad P_1(x)=x$$

for the first two Legendre polynomials. To obtain further terms, Eq. (1.4) is differentiated with respect to t on both sides and rearranged to give

$$\frac{x-t}{\sqrt{1-2xt+t^2}}=(1-2xt+t^2)\sum_{n=0}^{\infty}nP_n(x)t^{n-1}.$$

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Replacing the quotient of the square root with its definition in (1.4), and equating the coefficients of powers of t in the resulting expansion gives Bonnet's recursion formula:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x). \tag{1.5}$$

This relation, along with the first two polynomials P_0 and P_1, allows the Legendre polynomials to be generated recursively. The first few Legendre polynomials are

$$\begin{aligned} &P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \qquad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x, \\ &P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}, \qquad P_5(x) = \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x. \end{aligned} \tag{1.6}$$

In [2], the author considered the weighted heat equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[w(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x}\right] = -\lambda^2 v(x) \tag{1.7}$$

in the special case of

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}v(x)}{\mathrm{d}x}\right] = -\lambda^2 v(x). \tag{1.8}$$

Then Eq. (1.8) is Legendre's differential equation when $\lambda^2 = n(n+1)$.

Suppose that $v_{1,n} = P_n(x)$; then from the variation of parameters method, the additional solutions are given by the formula

$$v_{2,n} = P_n(x)q_n(x) \tag{1.9}$$

where

$$q_n(x) = \int \frac{\mathrm{d}x}{(1-x^2)P_n^2(x)}. \tag{1.10}$$

From (1.10), we have

$$q_0(x) = \int \frac{\mathrm{d}x}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|.$$

Then, by (1.9) we get

$$v_{2,0} = P_0(x)q_0(x) = \ln\sqrt{\left|\frac{1+x}{1-x}\right|}.$$

Using the same method, another three solutions are calculated:

$$v_{2,1} = x\ln\sqrt{\left|\frac{1+x}{1-x}\right|} - 1,$$

$$v_{2,2} = \frac{1}{2}(3x^2-1)\ln\sqrt{\left|\frac{1+x}{1-x}\right|} - \frac{3}{2}x$$

and

$$v_{2,3} = \frac{1}{2}(5x^3-3x)\ln\sqrt{\left|\frac{1+x}{1-x}\right|} - \frac{5}{2}x^2 + \frac{2}{3}.$$

Continued calculations carried out in a manner similar to those above would generate an infinite number of solutions to (1.8). But the integration in (1.10) becomes more and more complex as the index n increases.

In this work, we get a simple algorithm which allows us to do without the prohibitive integrations necessary to compute additional solutions.

2. The main result

Theorem 2.1. *The solutions of (1.8) satisfy the following relation:*

$$v_{2,n+1}(x) = \frac{2n+1}{n+1} x\, v_{2,n}(x) - \frac{n}{n+1}\, v_{2,n-1}(x). \tag{2.1}$$

Proof. We assume that the relation (2.1) is satisfied for $k = n$, that is

$$v_{2,n}(x) = \frac{2n-1}{n} x\, v_{2,n-1}(x) - \frac{n-1}{n}\, v_{2,n-2}(x). \tag{2.2}$$

Then, using (1.9), we have

$$nP_n(x)q_n(x) = (2n-1)xP_{n-1}(x)q_{n-1}(x) - (n-1)P_{n-2}(x)q_{n-2}(x). \tag{2.3}$$

If in (1.5) we decrease n by 1 we have

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x). \tag{2.4}$$

Substituting the recurrence relation (2.4) into Eq. (2.3) we find

$$(2n-1)xP_{n-1}(x)q_n(x) - (n-1)P_{n-2}(x)q_n(x) = (2n-1)xP_{n-1}(x)q_{n-1}(x) - (n-1)P_{n-2}(x)q_{n-2}(x). \tag{2.5}$$

If in (2.5) we increase n by 1 we get

$$(2n+1)xP_n(x)q_{n+1}(x) - nP_{n-1}(x)q_{n+1}(x) = (2n+1)xP_n(x)q_n(x) - nP_{n-1}(x)q_{n-1}(x). \tag{2.6}$$

By using (1.5) and (1.9) we can get

$$\begin{aligned} v_{2,n+1}(x) &= P_{n+1}(x)q_{n+1}(x) \\ &= \frac{2n+1}{n+1} xP_n(x)q_{n+1}(x) - \frac{n}{n+1} P_{n-1}(x)q_{n+1}(x). \end{aligned} \tag{2.7}$$

Substituting (2.6) into (2.7) we have

$$v_{2,n+1}(x) = \frac{2n+1}{n+1} xP_n(x)q_n(x) - \frac{n}{n+1} P_{n-1}(x)q_{n-1}(x). \tag{2.8}$$

Using (1.9) again we get the relation (2.1). □

Remark 2.1. It is interesting to note from the above theorem that each solution is dependent on the two preceding solutions. So we can compute the additional solutions of (1.8) easily; for example,

$$v_{2,4} = \frac{1}{8}(35x^4 - 30x^2 + 3)\ln\sqrt{\left|\frac{1+x}{1-x}\right|} - \frac{35}{8}x^3 + \frac{55}{24}x,$$

$$v_{2,5} = \frac{1}{8}(63x^5 - 70x^3 + 15x)\ln\sqrt{\left|\frac{1+x}{1-x}\right|} - \frac{63}{8} + \frac{49}{8}x^2 - \frac{8}{15},$$

etc.

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References

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[2] G.A. Bear, Legendre functions as solutions to the inhomogeneous heat equation, A Thesis in Mathematics, 1995.